Lecture 24:

Unsolvable Problems

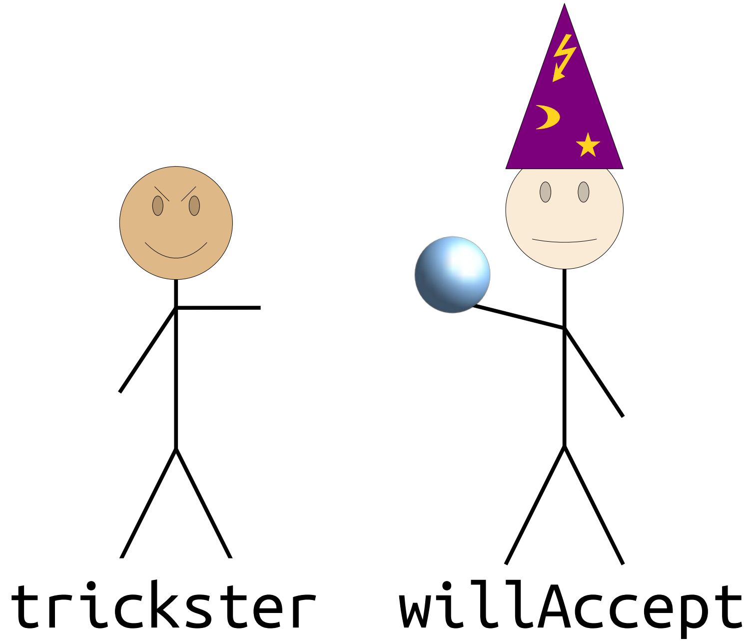
Part 2 of 2

Outline for Today

- ***More on Undecidability***
  - Even more problems we can’t solve.
- ***A Different Perspective on RE***
  - What exactly does “recognizability” mean?
- ***Verifiers***
  - A new approach to problem-solving.
- ***Beyond RE***
  - A beautiful example of an impossible problem.

Recap from Last Time

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns true.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```



Which of the following must be true?

(1) `trickster` is a decider for A_{TM}.

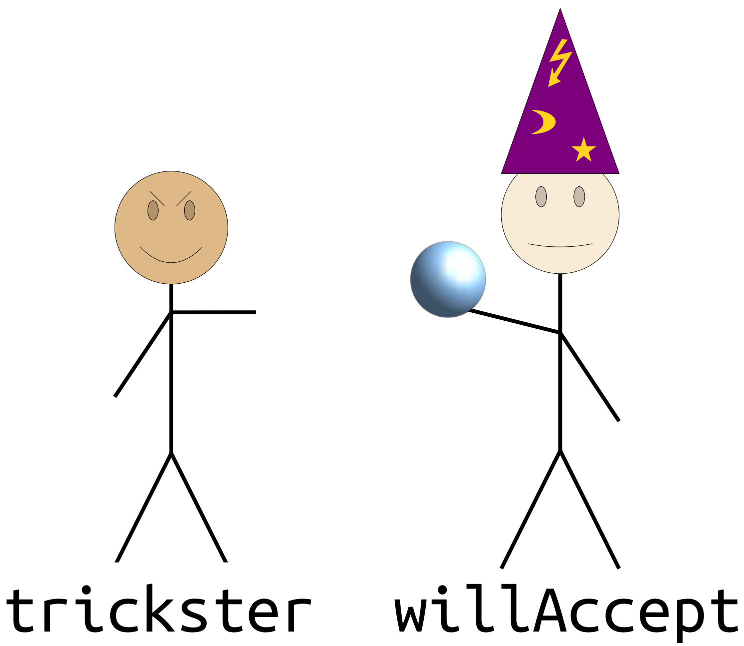
(2) `willAccept` is a decider for A_{TM}.

(3) `willAccept(me, input)` simulates `trickster` on `input` and does whatever `trickster` does to `input`.

(4) `trickster` loops on at least one input.

Answer at ***https://cs103.stanford.edu/pollev***

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns true.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```



A decider for A_{TM} has to have this behavior.

`trickster(input)` returns true

↔

`willAccept(me, input)` returns true

↔

`trickster(input)` does not return true

Because of how we wrote `trickster`.

Theorem: $A_{TM} \notin \mathbf{R}$.

Proof: By contradiction; assume that $A_{TM} \in \mathbf{R}$. Then there is a decider D for A_{TM}. We can represent D as a function

```
bool willAccept(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` returns true and returns false otherwise. Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```

Since `willAccept` decides A_{TM} and `me` holds the source of `trickster`, we know that

`willAccept(me, input)` returns true if and only if `trickster(input)` returns true.

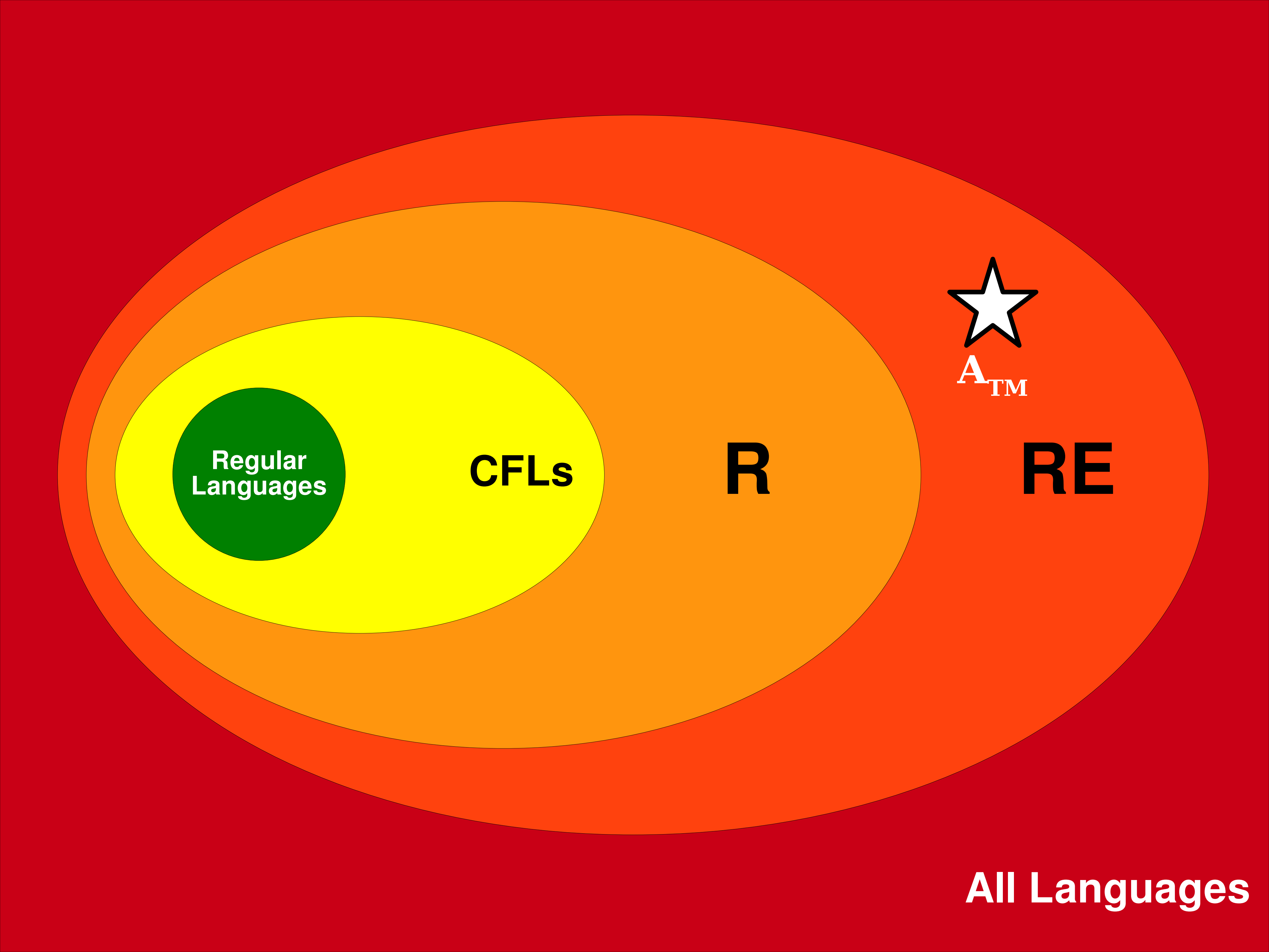
Given how `trickster` is written, we see that

`willAccept(me, input)` returns true if and only if `trickster(input)` doesn't return true.

This means that

`trickster(input)` returns true if and only if `trickster(input)` doesn't return true.

This is impossible. We've reached a contradiction, so our assumption was wrong and A_{TM} is undecidable. ■



Regular Languages
CFLs
R
RE
A_TM
All Languages

New Stuff!

More Impossibility Results

The Halting Problem

- The most famous undecidable problem is the ***halting problem***, which asks:

Given a TM *M* and a string *w*,
will *M* halt when run on *w*?

- Our goal isn't to build a TM *M* that halts on a string *w*. It's to check whether an arbitrary TM *M* halts on an arbitrary string *w*.
- As a formal language, this problem would be expressed as

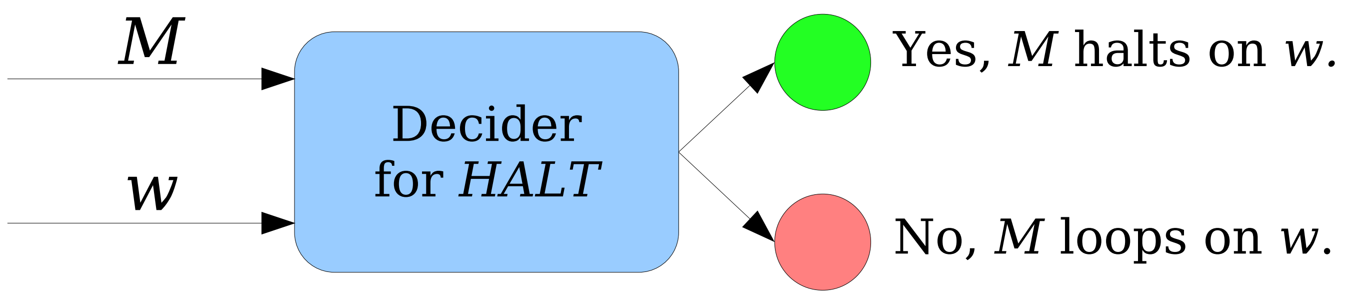
$$\boldsymbol{HALT = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on } w \}}$$

- ***Theorem:*** *HALT* is recognizable, but undecidable.
  - There's a recognizer for *HALT*.
  - There is no decider for *HALT*.

Theorem: The halting problem is undecidable.

A Decider for *HALT*

- Let's suppose that, somehow, we managed to build a decider for $HALT = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on } w \}$.
- Schematically, that decider would look like this:

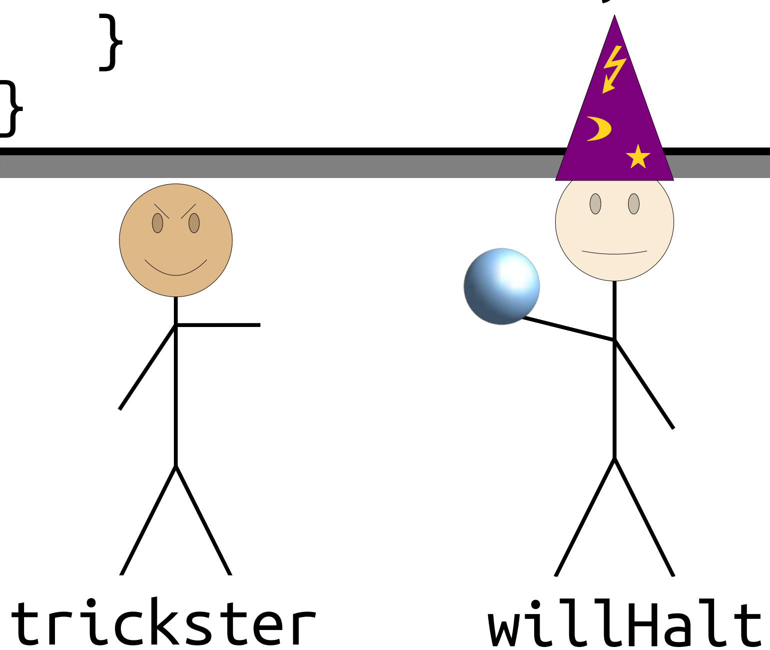


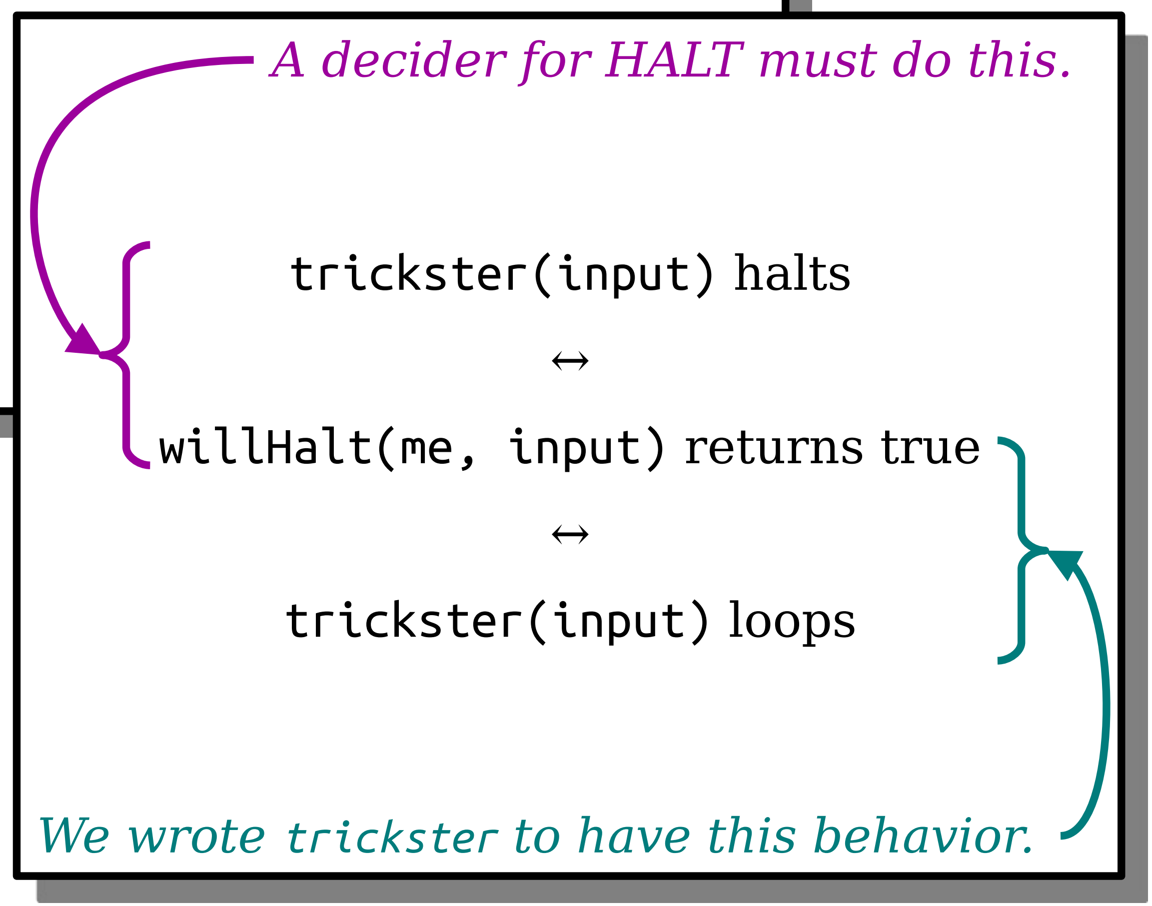
- We could represent this decider in software as a method

```
bool willHalt(string function, string input);
```

that takes as input a function `function` and a string `input`, then
  - returns true if `function(input)` returns anything (halts), and
  - returns false if `function(input)` never returns anything (loops).

```
bool willHalt(string function, string input) {
   // Returns true if function(input) halts.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;

   if (willHalt(me, input)) {
       while (true) {
           // Do nothing
       }
   } else {
       return true;
   }
}
```





Theorem: $HALT \notin \mathbf{R}$.

Proof: By contradiction; assume that $HALT \in \mathbf{R}$. Then there is a decider D for $HALT$. We can represent D as a function

```
bool willHalt(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` halts and returns false otherwise. Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (willHalt(me, input)) {
        while (true) { }
    } else {
        return true;
    }
}
```

Since `willHalt` decides $HALT$ and `me` holds the source of `trickster`, we know that

`willHalt(me, input)` returns true   if and only if   `trickster(input)` halts.

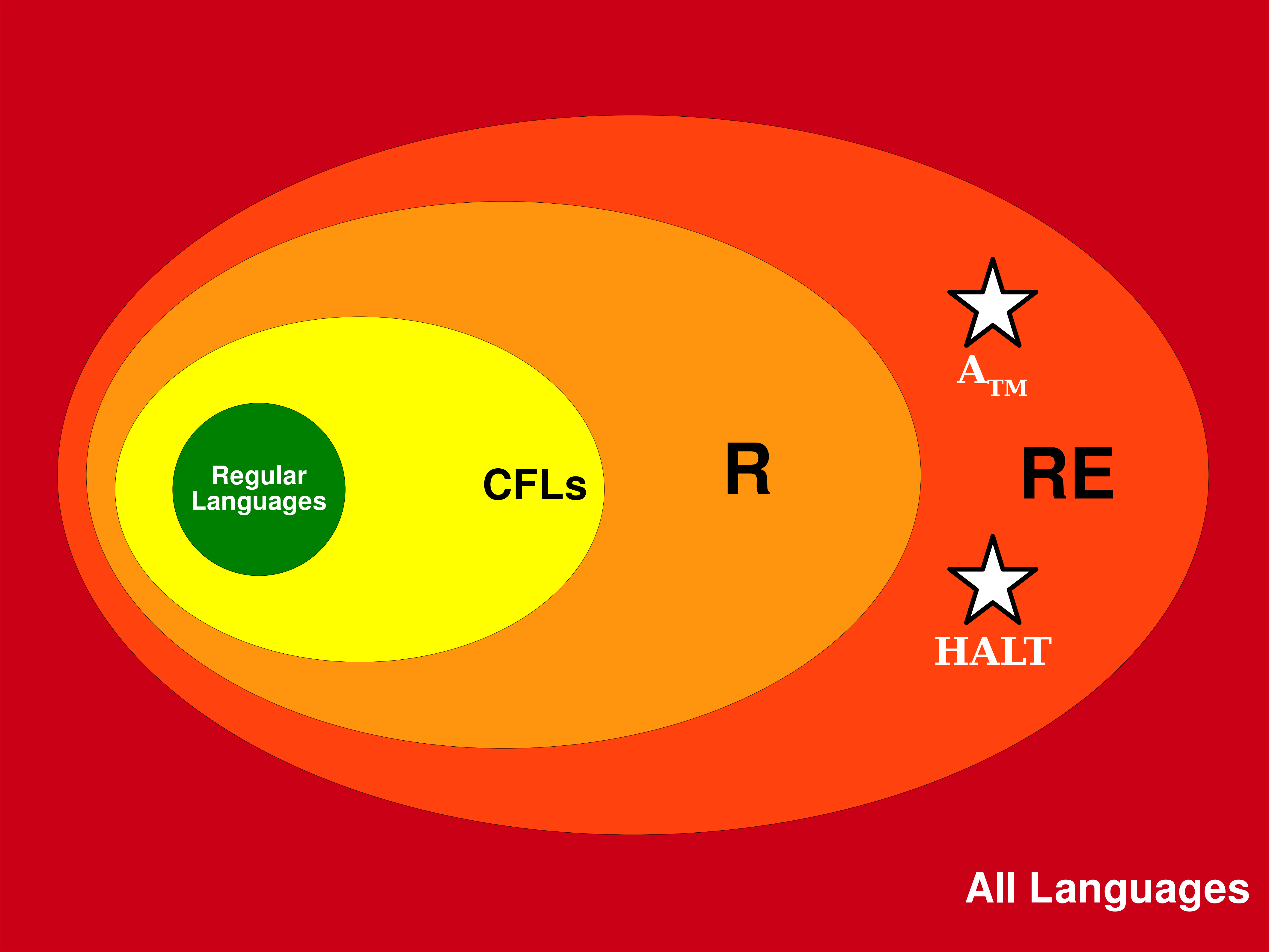
Given how `trickster` is written, we see that

`willHalt(me, input)` returns true   if and only if   `trickster(input)` loops.

This means that

`trickster(input)` halts   if and only if   `trickster(input)` loops.

This is impossible. We've reached a contradiction, so our assumption was wrong and $HALT$ is undecidable. ■

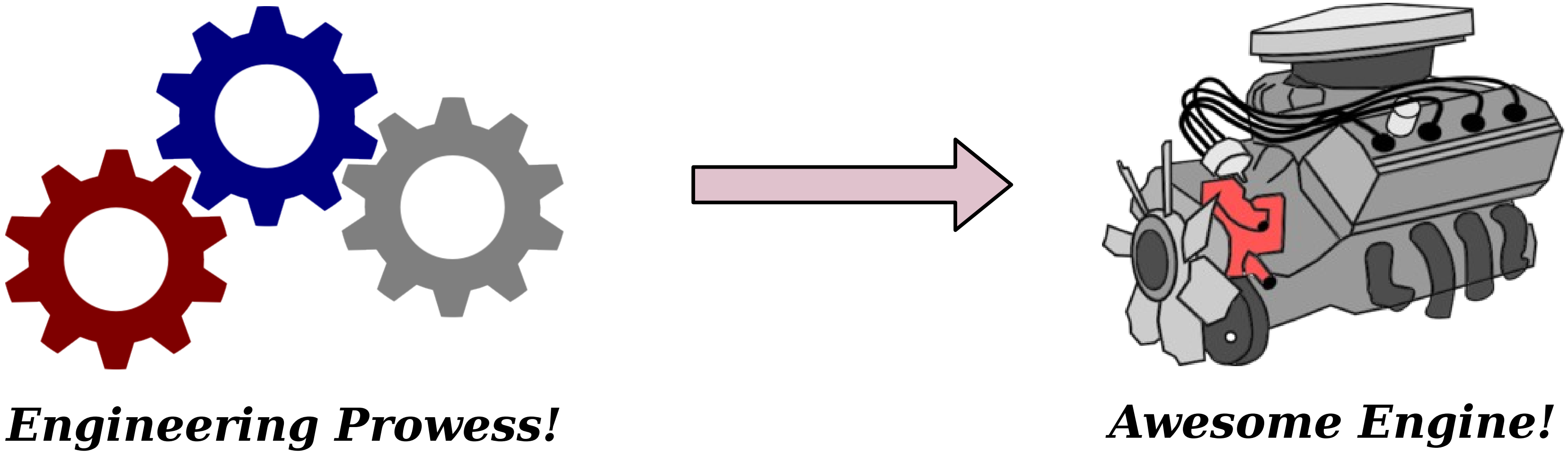

Regular Languages
CFLs
R
A_{TM}
RE
HALT
All Languages

So What?

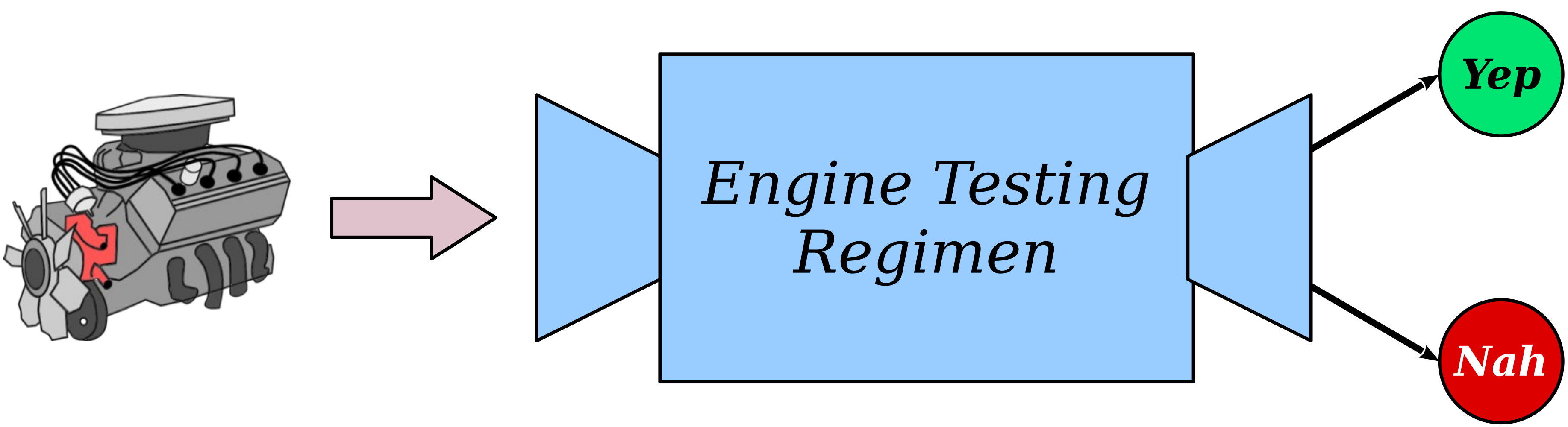
- These problems might not seem all that exciting, so who cares if we can't solve them?
- Turns out, this same line of reasoning can be used to show that some very important problems are impossible to solve.

Analogy Time!

Engineering Problem: Design a diesel engine that doesn't emit lots of NO_x pollutants.



Regulatory Problem: Design a testing procedure that, given a diesel engine, determines whether it emits lots of NO_x pollutants.



Fact: Almost all "regulatory problems" about computer programs are undecidable.

That is, almost all problems of the form "does program *X* have behavior *Y*?" are undecidable.

This can be formalized as ***Rice's Theorem***; take CS154 for details!

A (Topical) Example

Secure Voting

- Suppose that you want to make a voting machine for use in an election between two parties (the ***Zomp Party*** and the ***Puce Party***).
- Let $\Sigma = \{$`z`, `p`$\}$. A string $w \in \Sigma^*$ corresponds to a series of votes for the candidates.
- Example: `zzpppzp` means "two people voted for `z`, then three people voted for `p`, then one more person voted for `z`, then one more person voted for `p`."
- A ***secure voting machine*** is a TM that takes as input a string of `z`'s and `p`'s, then reports whether person `z` won the election.
  - "Secure" in the sense of "actually checks the vote totals" as opposed to rigging the election, discounting votes, etc.

A secure voting machine is a TM M where
M accepts $w \in \{$**z**, **p**$\}^*$ if and only if w has more **z**'s than **p**'s.

```
bool bee(string input) {
   int numZs = countZsIn(input);
   int numPs = countPsIn(input);

   return numZs > numPs;
}
```

```
bool topaz(string input) {
   return input != "" &&
          input[0] == 'z';
}
```

Which of these are secure voting machines? Answer at ***https://cs103.stanford.edu/pollev***

```
bool anna(string input) {
   int numZs = countZsIn(input);
   int numPs = countPsIn(input);

   if (numZs = numPs) {
      return false;
   } else if (numZs < numPs) {
      return false;
   } else {
      return true;
   }
}
```

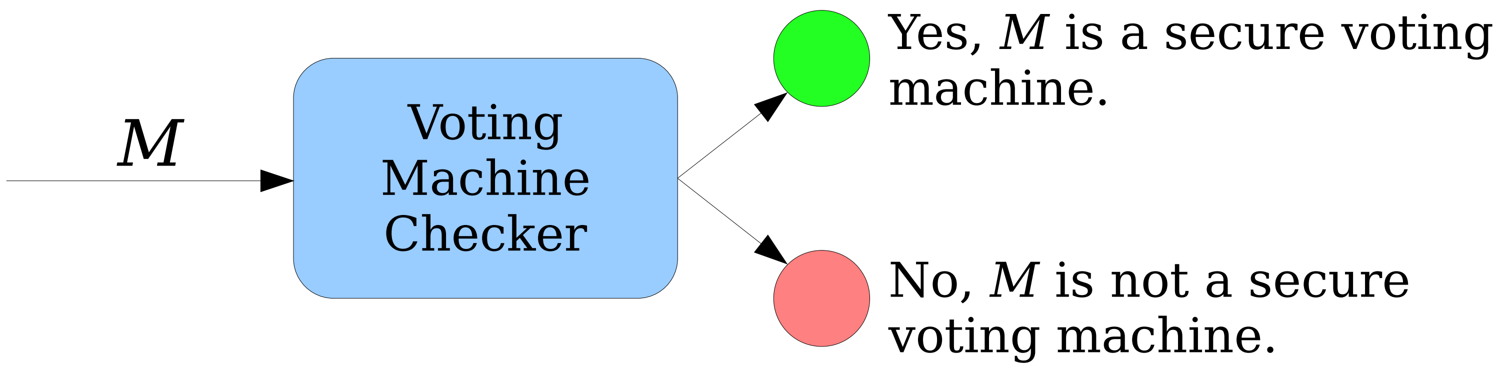
```
bool green(string input) {
   int n = input.length();
   while (n > 1) {
      if (n % 2 == 0) n /= 2;
      else n = 3*n + 1;
   }

   int numZs = countZsIn(input);
   int numPs = countPsIn(input);

   return numZs > numPs;
}
```

Secure Voting

- Even human review isn't perfect for vetting voting software.
- ***Question:*** Could we design an algorithm to check voting software for us?
  - ***Input:*** A Turing machine M.
  - ***Output:*** YES if M is a secure voting machine, NO if M isn't.
- This is a "regulatory" problem, not an "engineering" problem.

A Decider for Secure Voting

- Schematically, a "voting machine checker" would look like this:

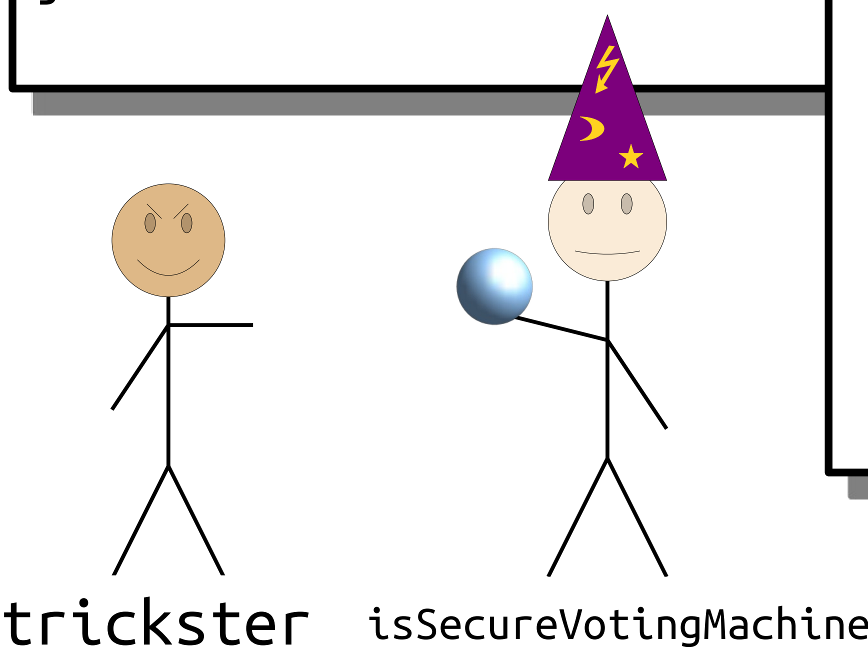


- We'd represent this decider in software as a function

```
bool isSecureVotingMachine(string function);
```

that takes as input a function, then returns whether that function is a secure voting machine.

```
bool isSecureVotingMachine(string function) {
   // Returns whether function accepts only
   // strings with more z's than p's.
}

bool trickster(string input) {
   string me = /* source code of trickster */;

   if (isSecureVotingMachine(me)) {
       return countZsIn(input) <= countPsIn(input);
   } else {
       return countZsIn(input) > countPsIn(input);
   }
}
```



`trickster` is a secure voting machine

$\leftrightarrow$

`isSecureVotingMachine(me)` returns true

$\leftrightarrow$

`trickster` isn't a secure voting machine.

Theorem: The secure voting problem is undecidable.

Proof: By contradiction; there is a decider *D* for the secure voting problem. We can represent *D* as a function

```
bool isSecureVotingMachine(string function);
```

that takes in the source code of a function `function`, then returns whether function is a secure voting machine (that is, whether it accepts precisely the strings with more `z`'s than `p`'s). Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    if (isSecureVotingMachine(me)) {
        return /* if input has at most as many z's as p's */;
    } else {
        return /* if input has more z's than p's */;
    }
}
```

Since `isSecureVotingMachine` decides the secure voting problem and `me` holds the source of `trickster`, we know that

`isSecureVotingMachine(me)` returns true if and only if `trickster` is a secure voting machine.

Given how `trickster` is written, we see that

`isSecureVotingMachine(me)` returns true if and only if `trickster` isn't a secure voting machine

This means that

`trickster` is a secure voting machine if and only if `trickster` isn't a secure voting machine.

This is impossible. We've reached a contradiction, so our assumption was and the secure voting problem is undecidable. ■

Interpreting this Result

- The previous argument tells us that *there is no automated procedure* that can check if arbitrary voting software is correct.
- So what can we do?
  - Design algorithms that work in *some*, but not *all* cases. (This is often done in practice.)
  - Fall back on human verification of voting machines. (We do that too.)
  - Carry a healthy degree of skepticism about electronic voting machines. (Then again, did we even need the theoretical result for this?)
- Worth a read: [***https://xkcd.com/2030/***](https://xkcd.com/2030/)

Time-Out for Announcements!

Problem Set 9

- Problem Set 8 is due ***Sunday*** at 1:00PM Pacific time.
  - You can use a late day to extend the deadline to Monday at 1:00PM Pacific.
  - If you're traveling, be cautious about time zone changes!
  - Late days can't be used on Problem Set 9.
- Problem Set 9 goes out today. It's due the Friday when we get back (December 5th).
  - This is a normally-sized problem set.
  - We are not expecting you to start it over the break.
  - Late days can't be used here; this is university policy.

Thanksgiving Break Logistics

- Over the break …
  - … we won't be holding regular office hours.
  - … we won't be monitoring EdStem as much as we normally do.
- ***International Students:*** If you've never attended an American Thanksgiving dinner, find a way to do so. It's a lovely tradition.
- ***Domestic, Local Students:*** If you're based locally and have the capacity to do so, invite a fellow student over for Thanksgiving.

Cumulative Practice Problems

- We've just released a ***massive*** bank of practice problems on the course website you can use to review topics from throughout the quarter.
- Feel free to ask us questions in office hours or on EdStem if you have them. That's what we're here for!
- Some post-midterm thoughts:
  - It's great to study this material and get practice. Just make sure to do it in a way that's maximally conducive to learning.
  - You're not competing against anyone else in this course. As you review for the final, form study groups. Share ideas and insights with one another.
  - We assign grades to certify skills, not based on relative performance. A's are not a scarce resource; we'd love to give as many as we can.
- Best of luck on the home stretch!

Back to CS103!

Beyond **R** and **RE**

What exactly is the class **RE**?

RE, Formally

- Recall that the class **RE** is the class of all recognizable languages:

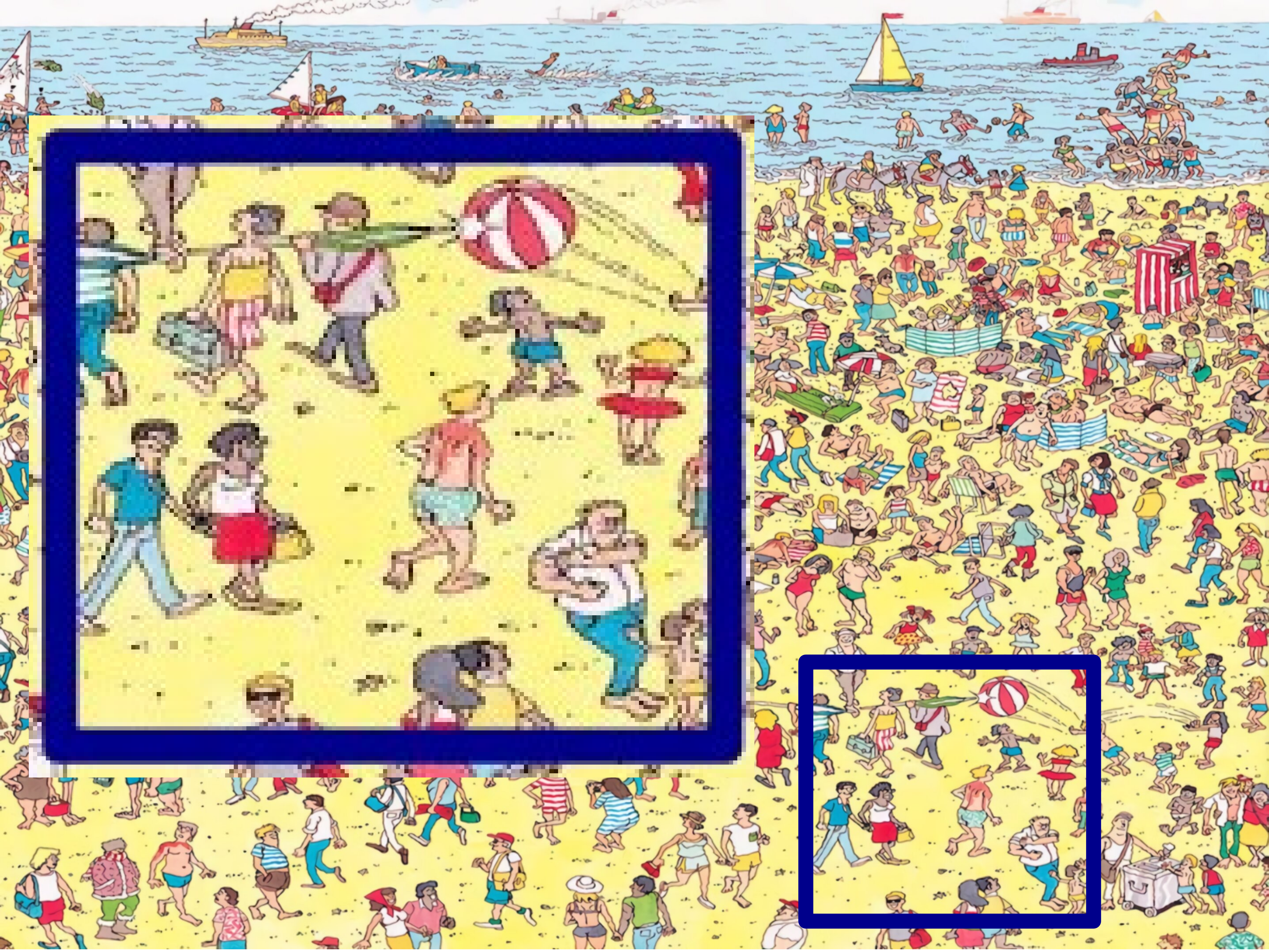
  RE = { L | there is a TM M that recognizes L }

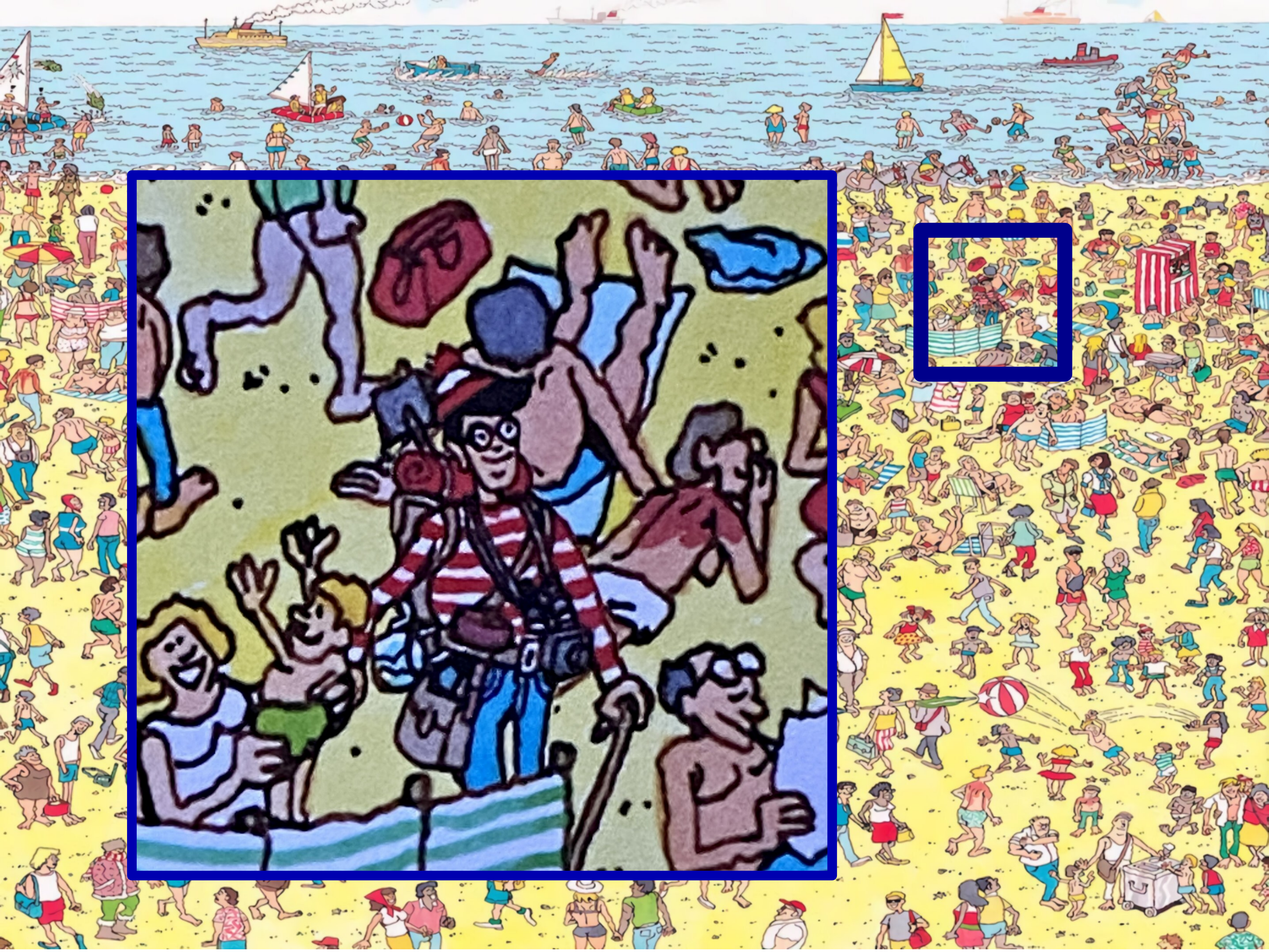
- Since **R** ≠ **RE**, there is no general way to "solve" problems in the class **RE**, if by "solve" you mean "make a computer program that can always tell you the correct answer."
- So what exactly *are* the sorts of languages in **RE**?

Key Intuition:

A language L is in **RE** when, for any string w, if you're *convinced* that $w \in L$, there's a way you could prove that to someone else.

Example: Where’s Waldo?





Verification

11

Try running five steps of the Hailstone sequence.

Does the hailstone sequence terminate for this number?

Verification

11

Try running fourteen steps of the Hailstone sequence.

Does the hailstone sequence
terminate for this number?

Verification

$$x^3 + y^3 + z^3 = 137$$

Pick the following:

$x = 3 \quad y = -5 \quad z = 6$

Are there integers x, y, and z where the above statement is true?

Verification

$$x^3 + y^3 + z^3 = 137$$

Pick the following:

x = -9 y = -11 z = 13

Are there integers x, y, and z where
the above statement is true?

Verification

- Here’s code for simulating the hailstone sequence. No one knows whether it always terminates.

```
bool hailstone(int n) {
    if (n <= 0) return false;
    while (n != 1) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
    }
    return true;
}
```

- The following doesn’t solve hailstone, but instead checks whether a given number of steps is correct. It always terminates.

```
bool checkHailstone(int n, int numSteps) {
    if (n <= 0) return false;
    for (int i = 0; i < numSteps; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
    }
    return n == 1;
}
```

Note the extra parameter.

Verification

- Here’s code that searches for three cubes that sum to a target. It loops if the *n* isn’t the sum of three cubes.

```
bool isCubeSum(int n) {
    for (int max = 0; ; max++)
        for (int x = -max; x <= max; x++)
            for (int y = -max; y <= max; y++)
                for (int z = -max; z <= max; z++)
                    if (x*x*x + y*y*y + z*z*z == n) return true;
}
```

- The following doesn’t solve the sum of cubes problems, but instead checks whether three numbers sum to the target. It always terminates.

```
bool checkCubeSum(int n, int x, int y, int z) {
    return x*x*x + y*y*y + z*z*z == n;
}
```

Note the extra parameters.

Verifiers

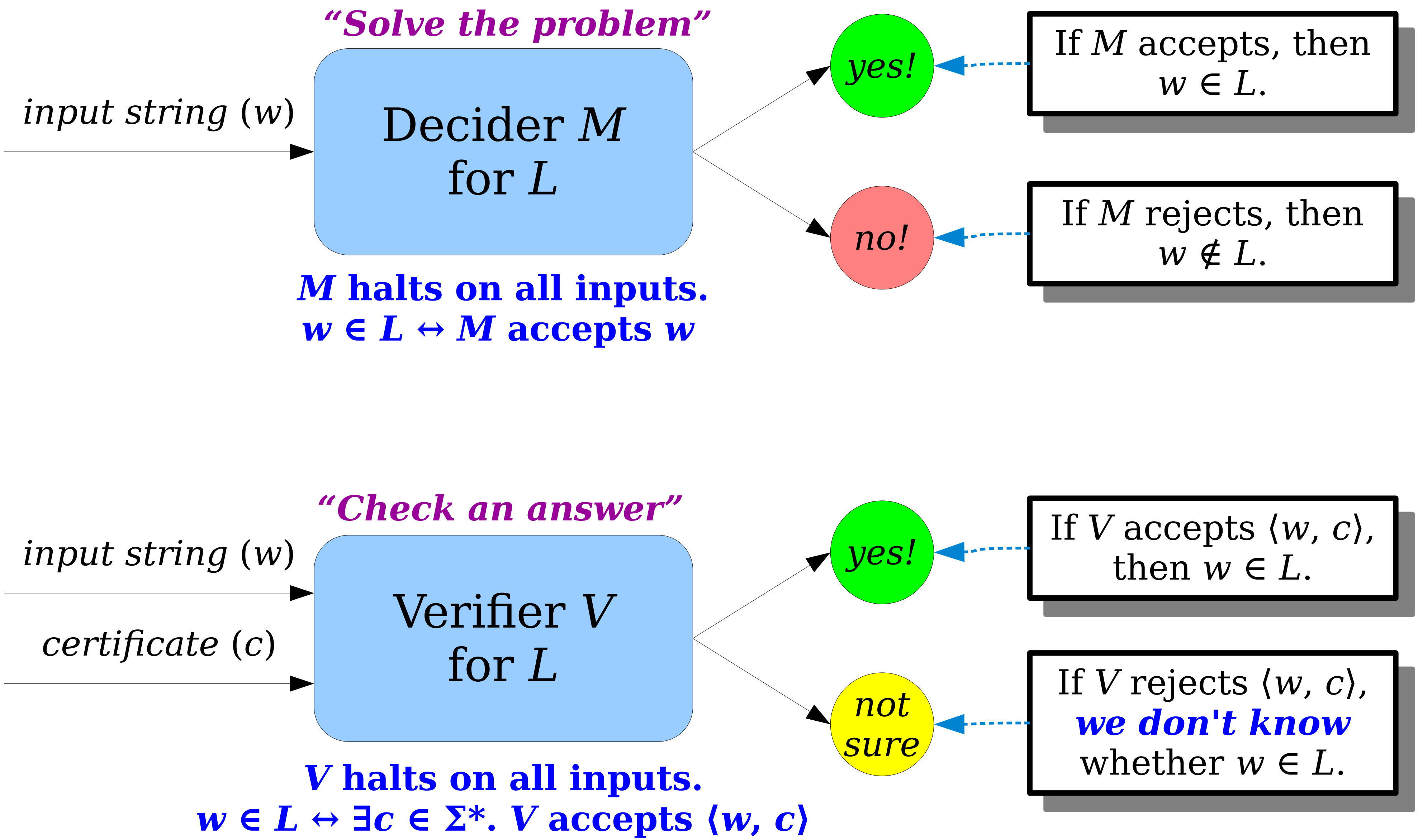
- A ***verifier*** for a language L is a TM V with the following two properties:

V halts on all inputs.

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$$

- Intuitively, what does this mean?

Deciders and Verifiers



Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:

V halts on all inputs.

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$$

- Some notes about V:
  - If V accepts $\langle w, c \rangle$, we're guaranteed $w \in L$.
  - If V rejects $\langle w, c \rangle$, then either
    - $w \in L$, but you gave the wrong c, or
    - $w \notin L$, so no possible c will work.

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:

V halts on all inputs.

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$$

- Some notes about V:
  - The certificate c is existentially-quantified. Any string $w \in L$ must have at least one c that causes V to accept, and possibly more.
  - V is required to halt, so given any potential certificate c for w, you can check whether the certificate is correct.

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:

V halts on all inputs.

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$$

- Some notes about V:
  - Although V always halts, V isn't a decider for L and isn't a recognizer for L. *(Do you see why?)*
  - V just checks certificates. It doesn't decide membership in L.

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:

V halts on all inputs.

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle)$$

- Some notes about V:
  - Remember that c can be an encoding of some other object or objects.
  - In practice, c will likely just be "some other auxiliary data that helps you out."

What languages are verifiable?

Theorem: If L is a language, then there is a verifier for L if and only if $L \in \mathbf{RE}$.

Proof: Appendix!

RE and Proofs

- Verifiers and recognizers give two different perspectives on the “proof” intuition for **RE**.
- A verifier V for L checks proofs that $w \in L$.
  - If $w \in L$, there’s a proof c where V accepts $\langle w, c \rangle$
  - If $w \notin L$, then V never accepts any certificate for w.
- A recognizer R for L searches for proof that $w \in L$.
  - If $w \in L$, then R finds a proof and accepts.
  - If $w \notin L$, then R never finds a proof and loops.
    - Or perhaps it finds a proof that $w \notin L$ and rejects.

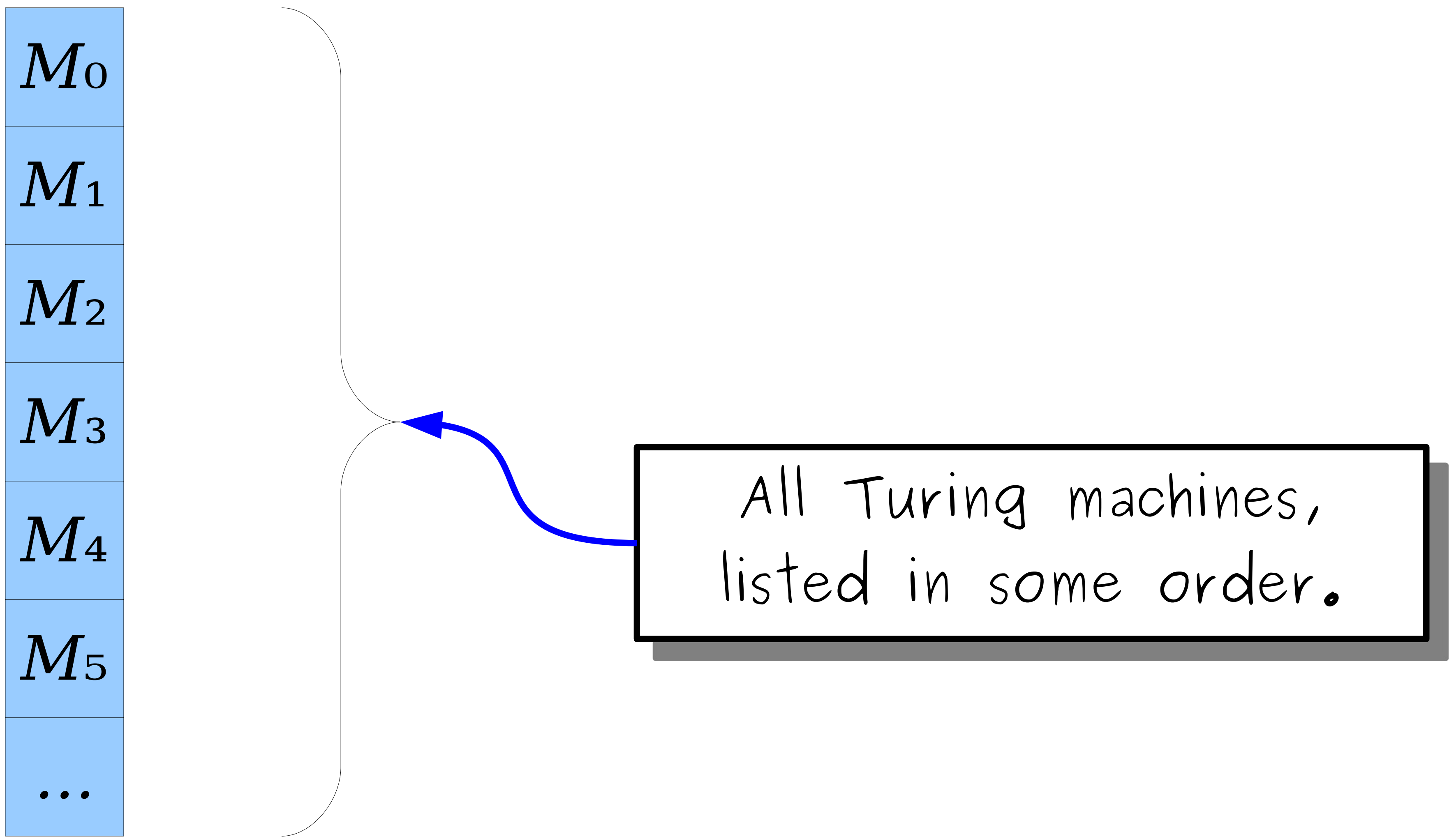
Finding Non-**RE** Languages

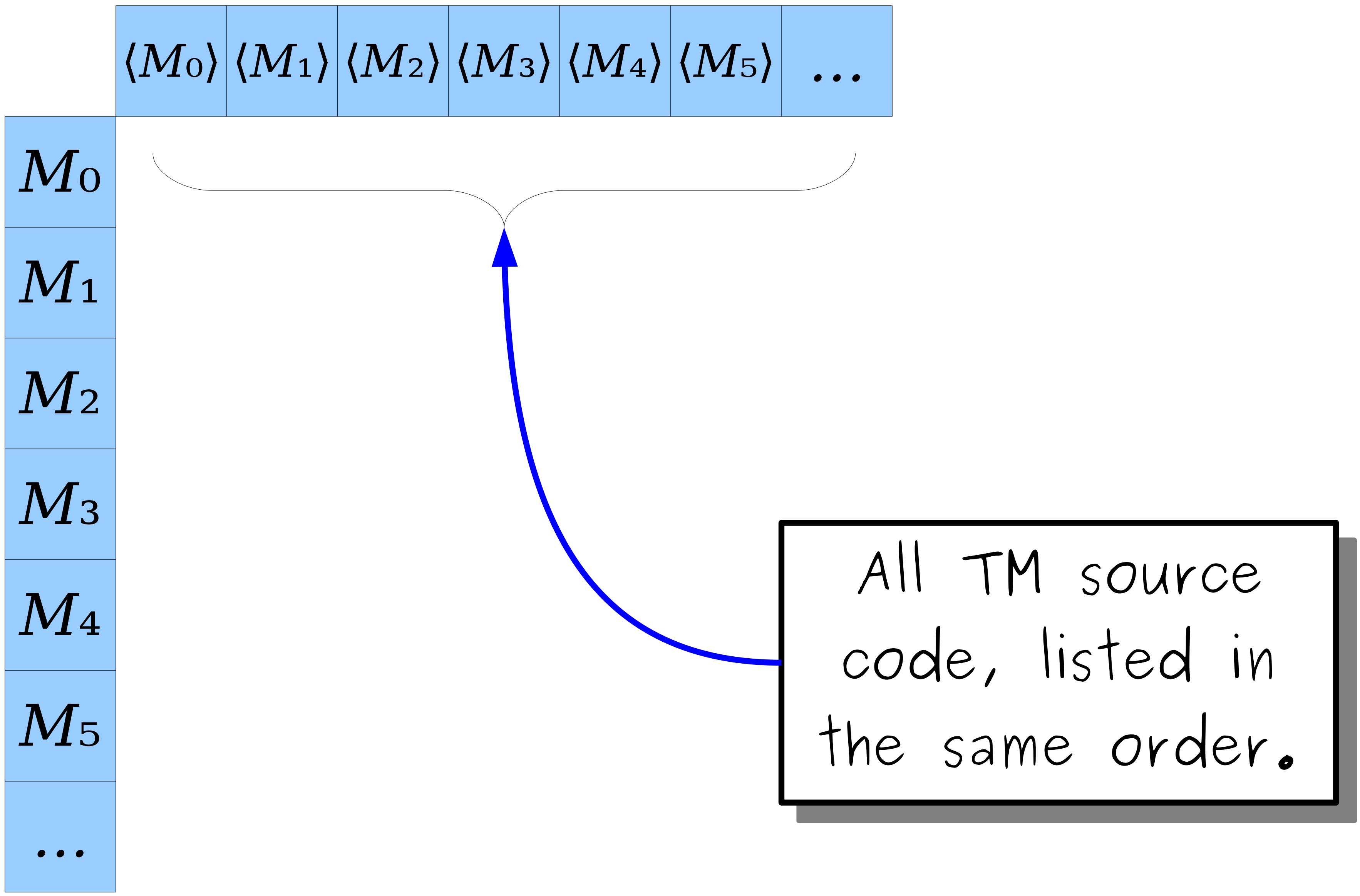
Recognizers and Recognizability

- ***Recall:*** We say that M is a recognizer for L if the following is true:

$$\forall w \in \Sigma^*. \ (w \in L \quad \leftrightarrow \quad M \text{ accepts } w).$$

- Some of these strings w, by pure coincidence, will be encodings of Turing machines.
- What happens if we list off all Turing machines, looking at how those TMs behave given other TMs as input?


M_0
M_1
M_2
M_3
M_4
M_5
...
All Turing machines, listed in some order.


⟨M0⟩ ⟨M1⟩ ⟨M2⟩ ⟨M3⟩ ⟨M4⟩ ⟨M5⟩ …
M0
M1
M2
M3
M4
M5
…
All TM source code, listed in the same order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	$\dots$
M_0	Acc	No	No	Acc	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	Acc	Acc	Acc	Acc	...
M_3	No	Acc	Acc	No	Acc	Acc	...
M_4	Acc	No	Acc	No	Acc	No	...
M_5	No	No	Acc	Acc	No	No	...
$\dots$	...	...	...	...	...	...	...
	Acc	Acc	Acc	No	Acc	No	...

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

Flip all "accept" to "no" and vice-versa

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

No TM has this behavior!

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

{ $\langle M \rangle$ | M is a TM that does not accept $\langle M \rangle$ }

Diagonalization Revisited

- The ***diagonalization language***, which we denote $\boldsymbol{L_{\mathrm{D}}}$, is defined as

$$\boldsymbol{L_{\mathrm{D}} = \{ \langle M \rangle \mid M \textbf{ is a TM and } M \textbf{ does not accept } \langle M \rangle \}}$$

- We constructed this language to be different from the language of every TM.
- Therefore, $L_{\mathrm{D}} \notin \mathbf{RE}$! Let's go prove this.

$L_D = \{ \langle M \rangle \mid M$ is a TM and M does not accept $\langle M \rangle \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: Assume for the sake of contradiction that $L_D \in \mathbf{RE}$. This means that there is a recognizer R for L_D.

What happens if we run R on $\langle R \rangle$? Since R recognizes L_D, we know that

$$R \text{ accepts } \langle R \rangle \qquad \text{if and only if} \qquad \langle R \rangle \in L_D.$$

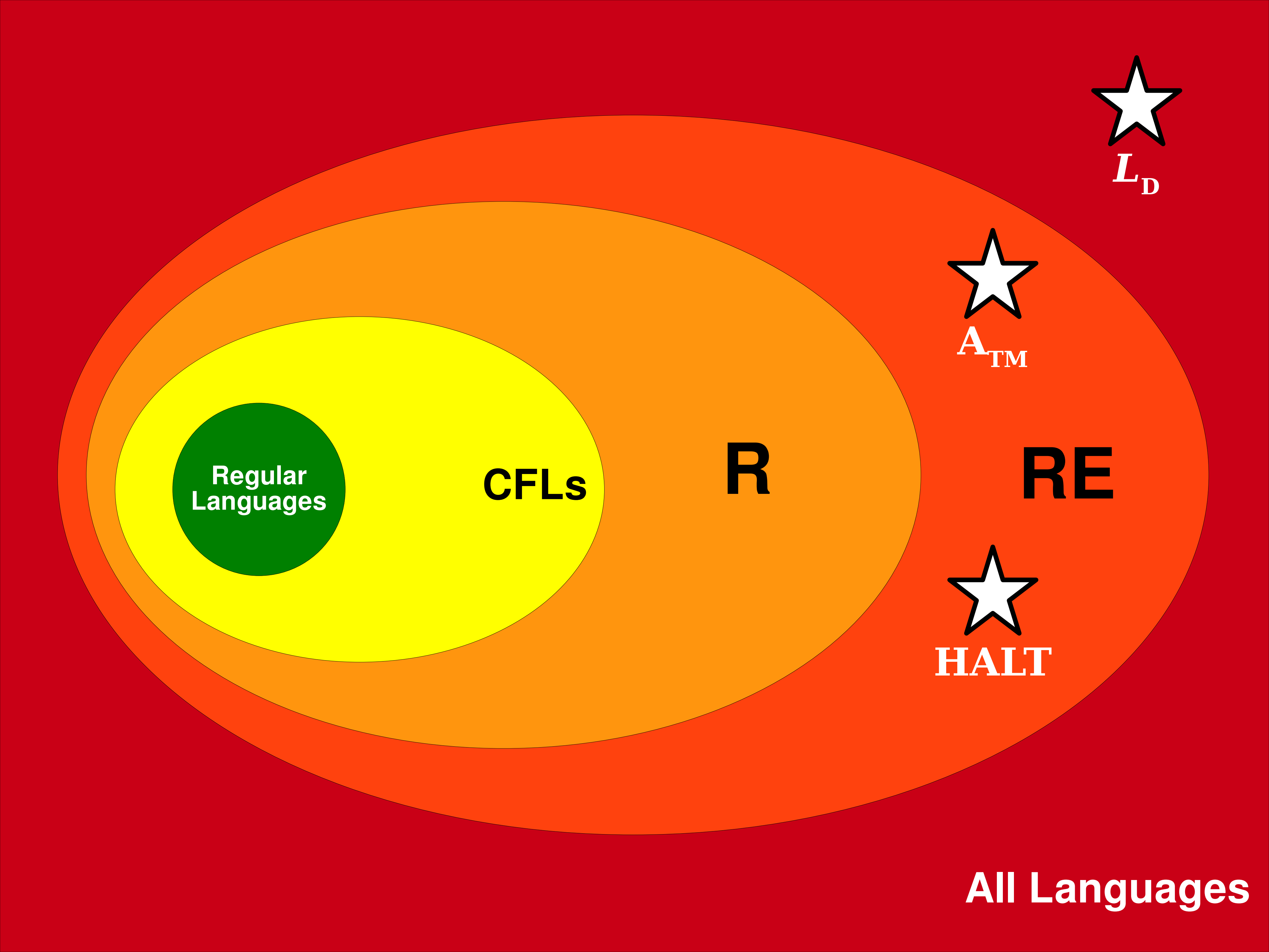
By definition of L_D, we also know that

$$\langle R \rangle \in L_D \qquad \text{if and only if} \qquad R \text{ does not accept } \langle R \rangle.$$

Combining the two above statements tells us that

$$R \text{ accepts } \langle R \rangle \qquad \text{if and only if} \qquad R \text{ does not accept } \langle R \rangle.$$

This is impossible. We've reached a contradiction, so our assumption was wrong, and so $L_D \notin \mathbf{RE}$. ■



L_D
A_{TM}
Regular Languages
CFLs
R
RE
HALT
All Languages

What This Means

- On a deeper philosophical level, the fact that non-**RE** languages exist supports the following claim:

  There are statements that are true but not provable.

- This result can be formalized as a result called ***Gödel's incompleteness theorem***, one of the most important mathematical results of all time.
- Want to learn more? Take Phil 152 or CS154!

What This Means

- On a more philosophical note, you could interpret the previous result in the following way:

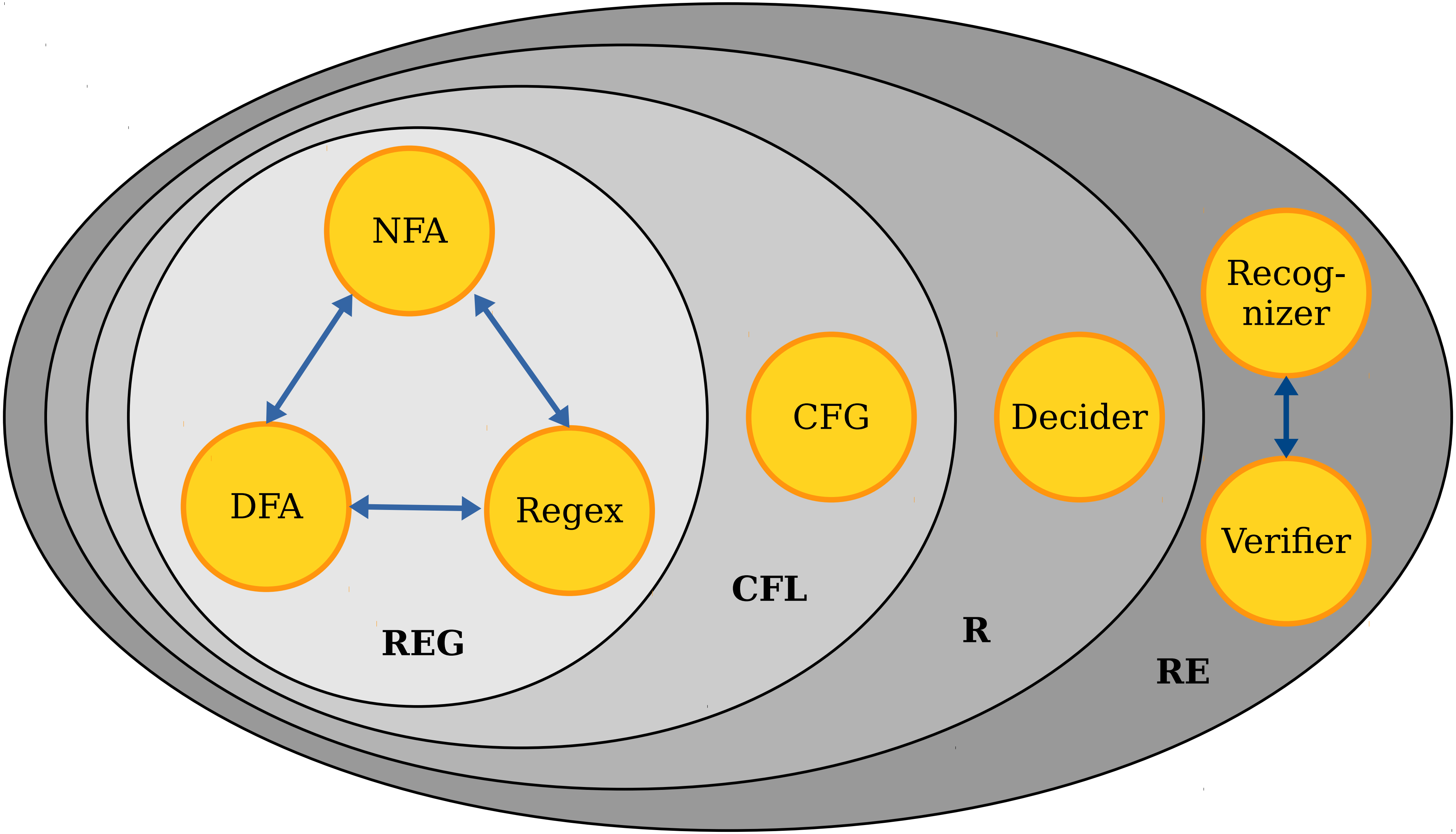
There are inherent limits about what mathematics can teach us.

- There's no automatic way to do math. There are true statements that we can't prove.
- That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.

Where We Stand

- We've just done a whirlwind tour of computability theory:
  - ***The Church-Turing thesis*** tells us that TMs give us a mechanism for studying computation in the abstract.
  - ***Universal computers*** – computers as we know them – are not just a stroke of luck. The existence of the universal TM ensures that such computers must exist.
  - ***Self-reference*** is an inherent consequence of computational power.
  - ***Undecidable problems*** exist partially as a consequence of the above and indicate that there are statements whose truth can't be determined by computational processes.
  - ***Unrecognizable problems*** are out there and can be discovered via diagonalization. They imply there are limits to mathematical proof.

The Big Picture



Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.

Next Time

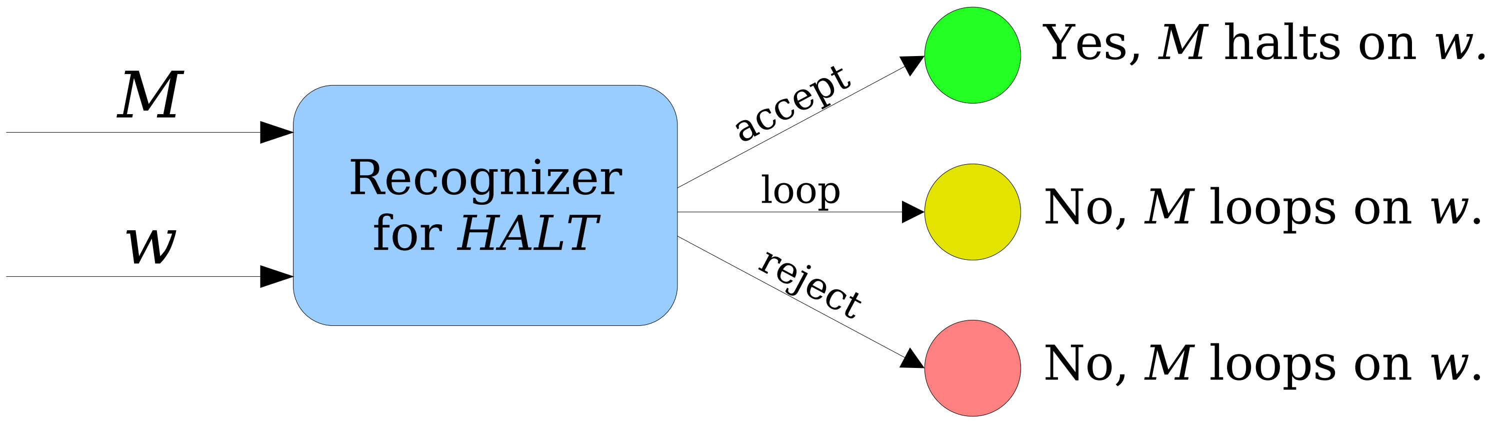
- ***Introduction to Complexity Theory***
  - Not all decidable problems are created equal!
- ***The Classes P and NP***
  - Two fundamental and important complexity classes.
- ***The $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ Question***
  - A literal million-dollar question!

Enjoy the Break!

Appendix 1: $HALT \in \mathbf{RE}$

HALT ∈ **RE**

- The halting problem is recognizable, meaning there's a recognizer for it.
- That recognizer would have the following abstract behavior:



$HALT \in \textbf{RE}$

- ***Idea:*** If you were certain that a TM *M* halted on a string *w*, could you convince me of that?
- Yes – just run *M* on *w* and see what happens!
- Here's that idea expressed as a recognizer:

```
bool recognizeIfHalts(string TM, string w) {
   set up a simulation of M running on w;
   while (true) {
      if (M returned true) return true;
      else if (M returned false) return true;
      else simulate one more step of M running on w;
   }
}
```

$HALT \in \mathbf{RE}$

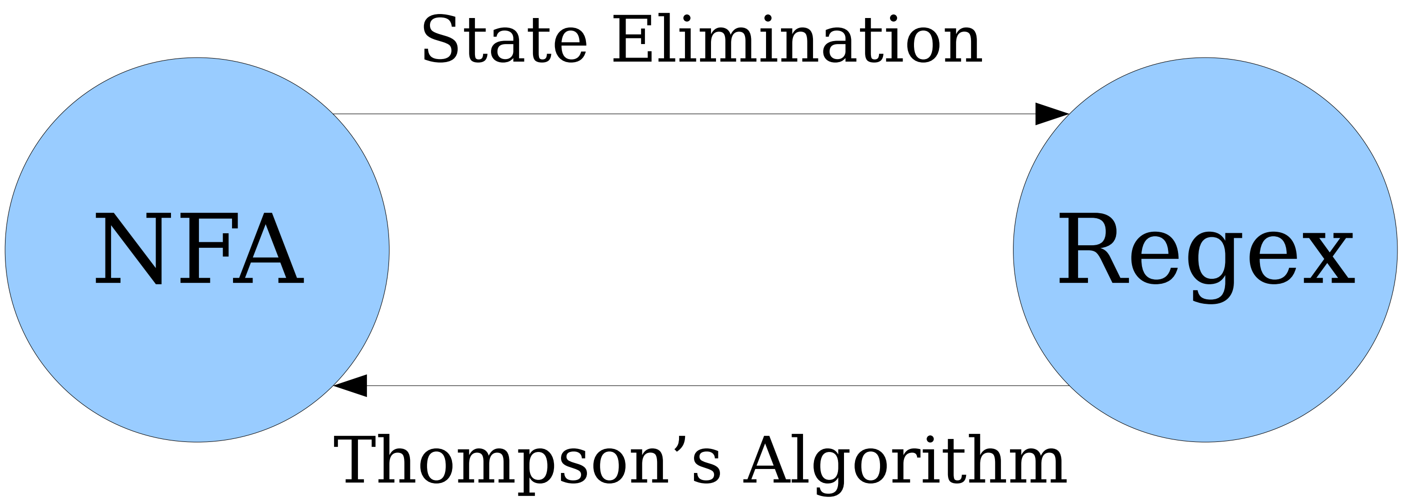
- How might we build a verifier for *HALT*?
- ***Idea:*** If a TM *M* halts on a string *w*, it must do so within some number of steps.
- Our verifier can then run *M* on *w* for that many steps and see if it halts:

```
bool checkAccepts(TM M, string w, int n) {
    set up a simulation of M running on w;
    for (int i = 0; i < n; i++) {
        simulate one more step of M running on w;
    }
    return whether M halted;
}
```

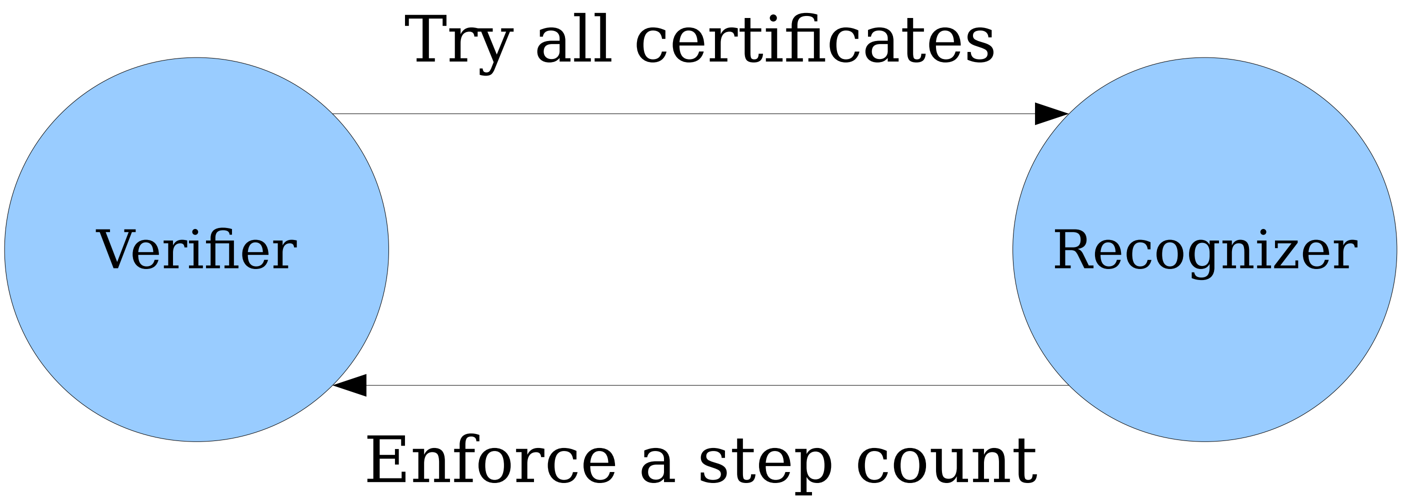
Appendix 2: Verifiers and **RE** Languages

Theorem: Let L be a language. Then $L \in \mathbf{RE}$ if and only if there is a verifier V for L.

Where We've Been



Where We're Going



Verifiers and **RE**

- ***Theorem:*** If V is a verifier for L, then $L \in$ **RE**.
- ***Proof sketch:*** Consider the following program:

```
bool isInL(string w) {
    for (each string c) {
        if (V accepts ⟨w, c⟩) return true;
    }
}
```

If $w \in L$, there is some $c \in \Sigma^*$ where V accepts $\langle w, c \rangle$. The function `isInL` tries all possible strings as certificates, so it will eventually find c (or some other working certificate), see V accept $\langle w, c \rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string c such that V accepted $\langle w, c \rangle$, so we see that $w \in L$. ■

Verifiers and **RE**

- ***Theorem:*** If $L \in \mathbf{RE}$, then there is a verifier for L.
- ***Proof sketch:*** Let L be a **RE** language and let M be a recognizer for it. Consider this function:

```
bool checkIsInL(string w, int c) {
    TM M = /* hardcoded version of a recognizer for L */;
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on W;
    }
    return whether M is in an accepting state;
}
```

Note that `checkIsInL` always halts, since each step takes only finite time to complete. Next, notice that if there is a c where `checkIsInL(w, c)` returns true, then M accepted w after running for c steps, so $w \in L$. Conversely, if $w \in L$, then M accepts w after some number of steps (call that number c). Then `checkIsInL(w, c)` will run M on w for c steps, watch M accept w, then return true. ■